SUBMERGED JETS ISSUING FROM CONICAL RADIALLY SLIT NOZZLES

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Self-similar flow within a jet domain as well as self-similar flow induced by jets in the surrounding medium are analyzed.

Experimental data presented in [1-3], as well as results of studies conducted by the author on special apparatus, show that submerged jets issuing from radially slit nozzles formed by two coaxial cones do not, by far, always take the shape of hollow conical jets. For example, in the case of propagation in an infinite space, a hollow conical jet is formed for half-angles greater than 60-65° at the nozzle exit. For lesser cone angles, a jet at the nozzle output closes to form a closed zone of circulation currents. A flow of the hollow conical jet type is generally never observed in the case of jet flow from a nozzle mounted in a wall, which is important in practice, but either a closing jet or a jet spreading along the wall bounding the stream is realized.

Conditions for propagation of laminar and turbulent hollow conical radially slit untwisted jets with a constant cone angle along the jet, for which jet equilibrium holds in a direction perpendicular to the main jet, are analyzed in this paper. The analysis is conducted for jets escaping from infinitesimal sources and, therefore, refers to real jets sufficiently remote from the nozzle sections.

Let an incompressible fluid jet escape from a nozzle formed by two infinitesimal funnels, one of which is imbedded in the other, and be propagated in the space bounded by a conical surface with a half-angle γ and filled with the same fluid. Let us select the spherical coordinates R, Θ, ε with polar axis ε along the axis of jet symmetry and origin at the point of jet emergence (Fig. 1).

Let us first consider a laminar jet. To do this, let us turn to the case, clarified by Slezkin [4], of exact integration of the motion equation of the steady axisymmetric viscous incompressible fluid, corresponding to the flow from a point pulsed source. In this case, the solution for the stream function ψ is sought in the form

$$\psi = \nu R f(\omega). \tag{1}$$

Here $\omega = \cos\Theta$ and the stream function is introduced in such a way that

$$U = \frac{1}{R^2 \sin\Theta}\frac{\partial\psi}{\partial\Theta},\quad V = -\frac{1}{R\sin\Theta}\frac{\partial\psi}{\partial R}. \tag{2}$$

Using the substitution

$$f = -2(1-\omega^2)\frac{d\ln F}{d\omega}$$

permits reducing the problem to integrating a linear second-order equation for the function F:

$$\frac{d^2F}{d\omega^2} + \frac{B_0 + B_1\omega + B_2\omega^2}{2(1-\omega^2)^2}F = 0. \tag{3}$$

The form of the pulse source can be made specific and the solution can be made to match the boundary conditions by an appropriate selection of the constants B_0, B_1, and B_2. In order to obtain the solution for a hollow conical jet, let us assume that the values of the constants B_0, B_1, and B_2 are connected by

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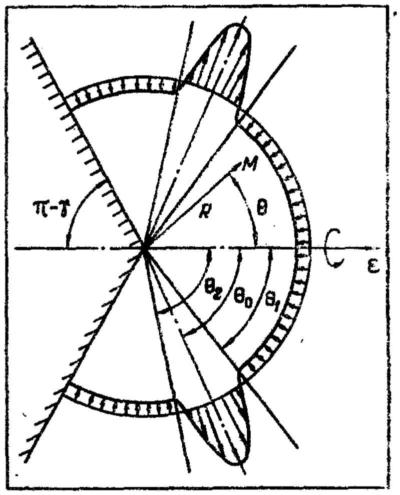

Fig. 1

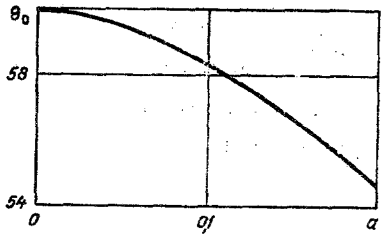

Fig. 2

Fig. 1. Diagram of a conical radially slit jet.

Fig. 2. Dependence of the self-similar jet half-angle on the parameter a for $\gamma = 90°$ (Θ_0, deg, and a dimensionless).

$$B_0 = B_2 = -\frac{1}{2} B_1 = 1 - \frac{b^2}{2}, \tag{4}$$

as in the case of the fan jet examined in [5]. Therefore, one constant, b, whose value is determined from the condition of conservation of the momentum flux, will enter into the solution.

Let us note that the half-angle of the jet Θ_0, determined by absence of a meridian velocity V on the ray $\Theta = \Theta_0$, does not vary along the jet because of self-similarity of the solution (1). Hence, the cases $\Theta_0 \leq 90°$ and $\Theta_0 < \gamma$ are considered. For a $\Theta_0 = 90°$ jet half-angle, the solution (1) should go over into the solution for the fan jet obtained in [5].

Taking account of (4), Eq. (3) is written as

$$\frac{d^2F}{d\omega^2} + \frac{1-b^2}{4(1+\omega)^2} F = 0. \tag{5}$$

Particular solutions of (5) are

$$F_1 = (1+\omega)^{\frac{1}{2}(1+b)}, \quad F_2 = (1+\omega)^{\frac{1}{2}(1-b)},$$

then

$$f(\omega) = (1-\omega)\left[-1 + b\,\frac{C_0 - (1+\omega)^b}{C_0 + (1+\omega)^b}\right], \tag{6}$$

where C_0 is a constant dependent on the boundary conditions.

Let us first examine the flow domain for $\Theta \leq \Theta_0$ and let us write the boundary conditions

$$f = 0 \text{ for } \omega = 1, \; f = 0 \text{ for } \omega = \omega_0 = \cos\Theta_0. \tag{7}$$

To comply with (7) it is necessary that

$$C_0 = (1+\omega_0)^b \frac{b+1}{b-1}.$$

We obtain the solution in the domain $\Theta \geq \Theta_0$ by replacing $(\omega - \omega_0)$ in (6) by $-(\omega - \omega_0)$. Consequently, we will have

$$f(\omega) = (1 + \omega - 2\omega_0)\left[-1 + b\,\frac{C_0 - (1 - \omega + 2\omega_0)^b}{C_0 + (1 - \omega + 2\omega_0)^b}\right]. \tag{8}$$

It follows from (8) that f = 0 for $\omega = 2\omega_0 - 1$, and by replacing the zero stream surface obtained by an impermeable wall, i.e., without taking account of the effect of fluid particles adhering to the wall surface, we obtain the relationship between the jet half-angle Θ_0 and the half-angle of the conical surface bounding the space in which the jet is propagated,

$$\cos\gamma = 2\cos\Theta_0 - 1, \tag{9}$$

which is needed for the existence of a self-similar solution.

Replacement of the physically correct condition of adhesion of a viscous fluid to a wall surface in the boundary condition from the theory of an ideal fluid should not essentially affect the result of a solution in the case of a jet with a sufficiently large Reynolds number, just as occurs in the problem of a jet being propagated within a cone along its axis [5].

In particular, condition (9) requires the value $\Theta_0 = 90°$ (fan jet) for the case of an infinite space ($\gamma = 180°$) and the value $\Theta_0 = 60°$ for the case of a semi-infinite space ($\gamma = 90°$).

Furthermore, let us examine a turbulent jet. The flow outside the turbulent region will be considered potential. Let us write the Reynolds equations in a spherical coordinate system taking account of axial symmetry and stationarity of the average flow and neglecting terms with molecular viscosity:

$$U\frac{\partial U}{\partial R} + \frac{V}{R}\frac{\partial U}{\partial \Theta} - \frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + \frac{1}{R^2}\frac{\partial}{\partial R}(R^2\langle -u^2\rangle) + \frac{1}{R\sin\Theta}\frac{\partial}{\partial\Theta}(\langle -uv\rangle\sin\Theta) - \frac{\langle -v^2\rangle + \langle -w^2\rangle}{R}, \tag{10}$$

$$U\frac{\partial V}{\partial R} + \frac{V}{R}\frac{\partial V}{\partial \Theta} + \frac{UV}{R} = -\frac{1}{\rho R}\frac{\partial p}{\partial \Theta} + \frac{1}{R^2}\frac{\partial}{\partial R}(R^2\langle -uv\rangle) + \frac{1}{R}\frac{\partial\langle -v^2\rangle}{\partial\Theta} + \frac{\langle -uv\rangle}{R} + \frac{\langle -v^2\rangle - \langle -w^2\rangle}{R}\operatorname{ctg}\Theta, \tag{11}$$

$$\frac{\partial}{\partial R}(R^2 U\sin\Theta) + \frac{\partial}{\partial\Theta}(RV\sin\Theta) = 0. \tag{12}$$

In addition to the continuity equation, the relationship

$$\frac{\partial U_p}{\partial\Theta} - \frac{\partial(RV_p)}{\partial R} = 0 \tag{13}$$

is satisfied in the potential flow region.

Let us seek the solution for the stream function ψ, introduced analogously to (2), in self-similar form within the turbulent domain:

$$\psi = ARf(\Theta), \tag{14}$$

where A is a constant along the jet which depends on the jet momentum. The projections of the average velocity are written in terms of the function f as

$$U = \frac{A}{R\sin\Theta}f', \quad V = -\frac{A}{R\sin\Theta}f. \tag{15}$$

Here and henceforth, the prime denotes differentiation.

By virtue of the assumption about self-similarity,

$$\langle -uv\rangle = \frac{A^2 g_0(\Theta)}{R^2}, \quad \langle -u^2\rangle = \frac{A^2 g_1(\Theta)}{R^2}, \quad \langle -v^2\rangle = \frac{A^2 g_2(\Theta)}{R^2}, \quad \langle -w^2\rangle = \frac{A^2 g_3(\Theta)}{R^2}, \quad \frac{p - p_\infty}{\rho} = \frac{A^2\Phi(\Theta)}{R^2} \tag{16}$$

Let us henceforth assume that $\langle -v^2 \rangle \approx \langle -w^2 \rangle$ by analogy with plane jets [6].

Substituting (15) and (16) into (11), we will have

$$\left[\Phi + \frac{1}{2}\left(\frac{f}{\sin\Theta}\right)^2 - g_2\right]' = g_0.$$

Integrating the relationship obtained across the jet while taking into account that g_2 vanishes on the jet boundaries, we obtain

$$\left[\Phi + \frac{1}{2}\left(\frac{f}{\sin\Theta}\right)^2\right]_{\Theta=\Theta_2} - \left[\Phi + \frac{1}{2}\left(\frac{f}{\sin\Theta}\right)^2\right]_{\Theta=\Theta_1} = \int\limits_{\Theta_1}^{\Theta_2} g_0 d\Theta, \tag{17}$$

where $\Theta_1 < \Theta_2$. The expression (17) is the equilibrium condition of a turbulent jet in the meridian direction. This condition can be obtained only on the basis of an unsimplified system of Reynolds equations in the boundary-layer assumptions.

For the potential domain in which the flow is also self-similar, there follows from (13) that

$$U_p = -\frac{C}{R}, \tag{18}$$

where the constant $C > 0$ from physical considerations. Substituting (18) into the continuity equation (12), we find

$$V_p = \frac{D - C\cos\Theta}{R\sin\Theta}. \tag{19}$$

The constants of integration D and C are determined from the boundary conditions separately for the domains $\Theta \leq \Theta_1$ and $\Theta \geq \Theta_2$.

By virtue of axial symmetry of the flow $V_p = 0$ for $\Theta = 0$ and from the condition of impenetrability of the cone surface $V_p = 0$ for $\Theta = \gamma$, then

$$D_i = C_i \cos\alpha_i, \tag{20}$$

where, for brevity, we have introduced $(i = 1, 2)$ $\alpha_1 = 0$, $\alpha_2 = \gamma$.

The condition for continuity of the projections of the velocity V on the jet boundaries requires that

$$\frac{C_i(\cos\alpha_i - \cos\Theta_i)}{R\sin\Theta_i} = -\frac{A}{R\sin\Theta_i} f(\Theta_i),$$

therefore,

$$C_i = -\frac{Af(\Theta_i)}{(\cos\alpha_i - \cos\Theta_i)}; \tag{21}$$

$$\Phi(\Theta_i) = \frac{R^2(p_i - p_\infty)}{A^2\rho} = -\frac{R^2(U_p^2 + V_p^2)_{\Theta=\Theta_i}}{A^2\rho} = -\frac{f^2(\Theta_i)}{2}\left[\frac{1}{\sin^2\Theta_i} + \frac{1}{(\cos\alpha_i - \cos\Theta_i)^2}\right]. \tag{22}$$

Substituting (22) into (17), we obtain the condition for jet equilibrium in the meridian direction as

$$\frac{f^2(\Theta_1)}{2(1-\cos\Theta_1)^2} - \frac{f^2(\Theta_2)}{2(\cos\gamma - \cos\Theta_2)^2} = \int\limits_{\Theta_1}^{\Theta_2} g_0 d\Theta. \tag{23}$$

Let us seek the function $f(\Theta)$ in the turbulent domain under the usual boundary-layer assumptions for an isobaric jet. Under these assumptions (10) becomes

$$U\frac{\partial U}{\partial R} + \frac{V}{R}\frac{\partial U}{\partial \Theta} = \frac{1}{R\sin\Theta}\frac{\partial}{\partial\Theta}(\langle -uv \rangle \sin\Theta). \tag{24}$$

Let us take as the boundary conditions for which we will solve (24):

$$U = 0 \text{ for } \Theta = \Theta_i, \quad \frac{\partial U}{\partial \Theta} = 0, \ V = 0 \text{ for } \Theta = \Theta_0. \tag{25}$$

Let us note that the radial velocity on the jet boundaries is not generally zero but is determined by (18) and (21). However, the quantity U_p is of the same order as the meridian velocity V_i on the jet boundaries, i.e., is a negligibly small quantity as compared with the maximal radial velocity in the jet.

To determine the quantity $\langle -uv \rangle$, let us take the hypothesis of a constant mixing path l on a surface of constant radius, where $l \sim R$. In this case we can write

$$\langle -uv \rangle = \varkappa \frac{\partial U}{\partial \Theta} \left| \frac{\partial U}{\partial \Theta} \right|. \tag{26}$$

Substituting (15) and (26) into (24), we will have

$$\varkappa \left[\left(\frac{f'}{\sin \Theta} \right)' \left| \left(\frac{f'}{\sin \Theta} \right)' \right| \sin \Theta \right]' + \left(\frac{ff'}{\sin \Theta} \right)' = 0. \tag{27}$$

Integrating (27) once and assuming on the basis of (25) that

$$\left(\frac{f'}{\sin \Theta} \right)' = 0 \text{ and } f = 0 \text{ for } \Theta = \Theta_0,$$

we obtain

$$\varkappa \left\{ \left[\left(\frac{f'}{\sin \Theta} \right)' \right]^2 \sin \Theta \right\} - (-1)^i \frac{ff'}{\sin \Theta} = 0. \tag{28}$$

Let us represent $\Theta = \Theta_0 + (-1)^i a\varphi$, where $a = \sqrt[3]{\varkappa}$, and let us rewrite (28) as

$$f'' - (-1)^i f' a \operatorname{ctg} [\Theta_0 + (-1)^i a\varphi] = -\sqrt{ff'} \tag{29}$$

for the domain $\Theta \leq \Theta_0$ by using the substitution $f \to -f$. The sign before the square root in the right side of (29) is selected from physical considerations about the nature of the profile $f'(\varphi)$.

According to (25), the boundary conditions for the function $f(\varphi)$ are the following

$$f = 0 \text{ for } \varphi = 0, \; f' = 0 \text{ for } \varphi = \varphi_i. \tag{30}$$

Moreover, by using the indeterminacy of the constant A up to this time, we assume

$$f' = 1 \text{ for } \varphi = 0. \tag{31}$$

By analogy with plane jets [7], it can be expected that $a \ll 1$, then approximately

$$a \operatorname{ctg} [\Theta_0 + (-1)^i a\varphi] \approx a \operatorname{ctg} \Theta_0 = \alpha. \tag{32}$$

It should be noted that the terms in (10) discarded under the boundary layer assumption contain the factor a^2.

Taking account of (32), Eq. (29) becomes

$$f'' - (-1)^i \alpha f' = -\sqrt{ff'}. \tag{33}$$

The solution of (33) for $\alpha = 0$ with the boundary conditions (30), (31) is presented in [7] and corresponds to a turbulent jet issuing from an infinitely long narrow slot.

Analogously to [7], let us introduce the new variable $z = \ln f(\varphi)$, then (33) goes over into the following equation for $z(\varphi)$:

$$z'' + (z')^2 - (-1)^i \alpha z' = -\sqrt{z'}.$$

Using the notation $z' = y^2$, we obtain an equation with separable variables $2y' + y^3 - (-1)^i \alpha y = -1$, whose solution is written as

$$\varphi = \text{const} - \frac{1}{[3\beta_i^2 - (-1)^i \alpha]} \left\{ \ln \frac{(y - \beta_i)^2}{[y^2 + \beta_i y + \beta_i^2 - (-1)^i \alpha]} - \frac{6\beta_i}{\sqrt{3\beta_i^2 - (-1)^i 4\alpha}} \operatorname{arctg} \frac{2y + \beta_i}{\sqrt{3\beta_i^2 - (-1)^i 4\alpha}} \right\}, \tag{34}$$

where β_i is the real root of the equation

$$y^3 - (-1)^i \alpha y + 1 = 0. \tag{35}$$

Because $\alpha \ll 1$ for not-too-small angles Θ_0, it can be assumed that

$$\beta_i \approx -[1+(-1)^i 0.333\alpha]$$

and by expanding (34) in a power series in α we will have the following to first-order accuracy:

$$\varphi = t_0(y) - (-1)^i 0.333\alpha\left[t_0(y) + \frac{2y^2}{y^3+1} + \frac{4}{\sqrt{3}}\operatorname{arctg}\frac{2y-1}{\sqrt{3}} - \frac{2\pi}{\sqrt{3}}\right]. \tag{36}$$

Here $t_0(y)$ is the solution (3) as $\alpha \to 0$ [7] and is determined by the following expression:

$$t_0(y) = \frac{\pi}{\sqrt{3}} - \frac{1}{3}\left[\ln\frac{(y+1)^2}{y^2-y+1} + \frac{6}{\sqrt{3}}\operatorname{arctg}\frac{2y-1}{\sqrt{3}}\right]. \tag{37}$$

Since $y=0$ for $f'=0$, it then follows from (36) that

$$\varphi_i = \frac{4\pi}{3\sqrt{3}}[1+(-1)^i 0.333\alpha]. \tag{38}$$

The solution (34) is not suitable near $\varphi=0$. It can be obtained that

$$z' = y^2 = \frac{1}{\varphi} + (-1)^i\alpha - 0.400\varphi^{1/2}, \quad z = \ln\varphi + (-1)^i\alpha\varphi - 0.266\varphi^{3/2}. \tag{39}$$

for small values of φ by the method elucidated in [7].

In combination with (39), the expression (36) permits finding the value of $f(\varphi)$ on the jet boundaries:

$$f(\varphi_i) = 0.996[1+(-1)^i 0.522\alpha]. \tag{40}$$

Let us evaluate the integral in the equality (23). It follows from (26) and (29) that to first-order accuracy

$$g_0 = -(-1)^i[1-(-1)^i 2\alpha\varphi](f^2)'(\sin\Theta_0)^{-2}.$$

Then to the same accuracy

$$\int_{\Theta_1}^{\Theta_2} g_0 d\Theta = (\sin\Theta_0)^{-2}\left[f^2(\varphi_1) - f^2(\varphi_2) + 4\alpha\int_0^{\varphi_0}(f_0^2)'\varphi d\varphi\right] = (\sin\Theta_0)^{-2}[f^2(\varphi_1) - f^2(\varphi_2) + 5.604\alpha]. \tag{41}$$

The function $f_0(\varphi)$ and the value of φ_0 correspond to (37) [7].

Substituting (38), (40), (41) into the jet equilibrium condition (23), we obtain the relationship between the conical surface half-angle and the jet half-angle which is necessary for the self-similar solution

$$\frac{(1-1.044a\operatorname{ctg}\Theta_0)\sin^2\Theta_0}{[1-\cos(\Theta_0-2.412a)]^2} - \frac{(1+1.044a\operatorname{ctg}\Theta_0)\sin^2\Theta_0}{[\cos\gamma-\cos(\Theta_0+2.412a)]^2} = 7.032a\operatorname{ctg}\Theta_0 \tag{42}$$

to the same accuracy.

The quantity a enters as a parameter into the relationship obtained between γ and Θ_0.

Let us examine the case $\gamma = 90°$ in more detail. Represented in Fig. 2 is a curve characterizing the change in the angle Θ_0 as a function of the value of the parameter a. It is seen that an increase in a corresponding to a rise in the degree of turbulence in the jet will cause a decrease in the quantity Θ_0.

Experiments show that the equilibrium of a hollow conical jet is not stable in the case $\gamma = 90°$. In this connection, the jet considered is a quasistationary model of a real jet in the initial instant of escape. It can be assumed that jets with a half-angle less than Θ_0 in sections sufficiently remote from the nozzle at this time, will be closed, while otherwise a jet spreading along the wall will be realized.

In conclusion, let us note that at short ranges from the nozzle, i.e., in the non-self-similar flow domain, some increase in the jet half-angle will occur, hence, the critical angle of a conical radially slit nozzle separating the case of a closing jet and a jet spreading along the wall turns out to be somewhat less than the angle Θ_0. Thus, according to [2], the critical angle is close to 50° for a nozzle with the ratio $2s/d = 0.17$, where s is the slot width and d is the outer diameter of the nozzle exit section.

NOTATION

a, empirical constant for a turbulent jet; A, constant determined by the turbulent jet momentum; b, constant associated with the laminar jet momentum; B_0, B_1, B_2, C, C_0, C_i, D_i, constants of integration; l, mixing path length; p, pressure; p_∞, pressure in the space into which the jet emerges; R, Θ, ε, spherical coordinate system; U, V, W, velocity components along the R, Θ, ε axes in the turbulent domain, averaged with respect to the time; u, v, w, pulsating velocity components; U_p, V_p, velocity components in the potential domain; α, small parameter; γ, half-angle of a conical surface; Θ_0, jet half-angle; Θ_1, Θ_2, angles corresponding to the turbulent domain boundaries; $\varkappa$, empirical constant; ν, coefficient of kinematic fluid viscosity; ρ, fluid density; ψ, stream function; $i = 1, 2$, domain with $\Theta < \Theta_0$ and domain with $\Theta > \Theta_0$, respectively; $\langle \rangle$, sign of averaging with respect to time.

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